THE ANALYTICAL STRUCTURE OF THE PRIMARY INTERSTELLAR HELIUM DISTRIBUTION FUNCTION IN THE HELIOSPHERE

Short Title: Analytical Structure of Interstellar Helium

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ABSTRACT

A new analytical model based on the previous work of Lee et al. (2012) is presented for the distribution of interstellar helium in the heliosphere. The model is tailored for comparison with the IBEX-Lo observations in order to determine the bulk velocity and temperature of helium in the local interstellar cloud. The model includes solar gravity, spherically-symmetric stationary ionization rates, transformation to the Earth/IBEX frame of reference, the IBEX viewing geometry with small spin axis tilt, and integration of the atom differential intensity over energy and the instrument collimator solid angle. The analysis employs an expansion of the count rate about the peak of the velocity distribution to second order in the magnitudes of several small quantities: the ratio of the helium thermal speed to its bulk speed, the angle between the bulk velocity and the ecliptic, the two angles describing the tilt of the IBEX spin axis away from Sun pointing, the collimator angular width, and the angular difference between the observing longitude and the longitude where the projection of the bulk velocity onto the ecliptic is tangential to Earth’s orbit. The model reveals the evolving ellipsoidal shape of the helium distribution as it moves along its average hyperbolic orbit. For specified interstellar parameters, the model predicts the latitudinal and longitudinal structure of the helium distribution. The model is in reasonable agreement with IBEX observations and the predictions of the other available models.

*Key words*: ISM: atoms and abundances – local interstellar medium – solar neighborhood – Sun: heliosphere

1. INTRODUCTION

The interstellar neutral gas blows into the heliosphere as the Sun travels through the local interstellar medium. The basic interaction is stationary and simple: the individual atoms are deflected by the Sun’s gravitational field and ionized at a rate proportional to , where is heliocentric radial distance. For hydrogen, UV radiation pressure partially counteracts the gravitational attraction.



**The resulting distribution of interstellar gas in the heliosphere was investigated in the 1960s and 1970s.** **Early models assumed that the interstellar gas is cold (Fahr, 1968; Blum and Fahr, 1970; Axford, 1972; Wallis, 1973). For this case, with the exception of the line parallel to the interstellar gas velocity extending downstream from the center of the Sun, which is traversed by trajectories from all directions around the Sun, every spatial location is traversed by two gas trajectories: the direct and indirect trajectories. Away from this downstream line of symmetry, the indirect trajectory generally passes close to the Sun and is substantially depleted by ionization losses. The continuous multiplicity of trajectories along the downstream line of symmetry is due to the focusing of the atom trajectories by solar gravitation. Based on the cold model, Vasyliunas and Siscoe (1976) predicted the phase-space distribution of interstellar pickup protons in the solar wind due to charge exchange between solar wind protons and interstellar hydrogen, assuming that the pickup protons are rapidly scattered in pitch angle. Fahr (1971, 1979), Thomas (1978), and Wu and Judge (1979) investigated the heliospheric distribution of interstellar gas for finite temperature. Since this case is analytically more challenging, these treatments were limited either to the downwind focused atoms, calculation of the number density, or the variation of the velocity distribution along a radial line. Müller (2012) recently calculated numerically the distribution of interstellar helium at arbitrary locations in the heliosphere. Zank (1999) provides an excellent review of the interaction between the interstellar neutral gas and the heliosphere.**

In spite of the underlying simplicity, there are complications in the basic interaction **of the interstellar neutral gas with the heliosphere** that may yield measureable effects. The ionization rates of the interstellar gas **by photo-ionization, charge exchange and electron impact** vary spatially and temporally **due to the UV line shapes and** the **variation of the** solar wind and UV intensities **(e.g. Scherer et al., 2014; Sokol et al., 2015)**. The individual atoms also suffer infrequent deflections that modify their ideal hyperbolic trajectories. The IBEX-Lo measurement technique, the geometry, and the reference frame of the measurement introduce further complications. Atoms are constrained to enter the instrument through a collimator of fixed solid angle. For helium the energy of the atom is determined primarily by measuring the negatively charged hydrogen sputtered at the conversion surface with a broad distribution in energy. The bore sight of the collimator scans perpendicular to the spacecraft spin axis, which is tilted by small angles from the Sun-pointing direction. Although the atom velocity distribution is most easily derived in the inertial frame of the Sun, the measurements are made in the IBEX frame, which varies over the IBEX orbit and is only approximately the frame of the Earth. Finally, Earth’s orbit is only approximately a circle about the Sun.

In order to infer the bulk velocity and temperature of an interstellar gas species in the local interstellar cloud from IBEX observations, the measurements must be compared with a theoretical model that includes the basic interaction, the transformation to the IBEX frame, the measurement technique, and the IBEX viewing geometry, including, in principle, those complications enumerated above that produce measureable systematic effects. **The Warsaw Group (**Bzowski et al., 2012**; Sokol et al., 2015)** has developed a numerical test particle model, which includes essentially all features discussed above, including atom deflection, time-dependent ionization, the sputtered ion distribution in energy, the motion of IBEX relative to Earth, and the eccentricity of Earth’s orbit. However, the comprehensiveness of this model has made the results difficult to interpret and check. As a result Lee et al. (2012) developed an analytical model based on Liouville’s Theorem for the basic stationary interaction, which includes transformation to Earth’s frame, and integration of atom intensity over atom speed and the solid angle aperture of the collimator. However, it also assumes that the collimator bore sight is perpendicular to the Sun-Earth line and that the ecliptic longitude of observation is close to the optimal longitude (the “sweet spot”) at which the bulk velocity of the interstellar gas is parallel to the bore sight at the latitude of maximum intensity. Both the **Warsaw** numerical models **(**Bzowski et al., 2012**; Sokol et al., 2015)** and the analytical model of Lee et al. (2012) assume that the atom distribution in the local interstellar cloud is a drifting Maxwellian distribution.

The restriction of the analytical model of Lee et al. (2012) to bore sights perpendicular to the Sun-Earth line makes it impossible for the model to incorporate a tilt of the bore sight out of the ecliptic or to describe effects due to the drift of the spin axis in the ecliptic away from Sun pointing as the Earth orbits the Sun. Inclusion of the spin axis tilt out of the ecliptic is essential since it was implemented during several IBEX orbits to provide a more accurate determination of the inflow longitude of the interstellar gas (Möbius et al., 2012). Inclusion of the drift of the spin axis in the ecliptic is also essential since the spacecraft can only be realigned once or twice in each orbit about the Earth, which implies a spin axis drift away from Sun pointing by up to at least 3 - 4 degrees at some phase in the orbit (McComas et al., 2009, 2011). In addition, the restriction in the model to atoms observed near their perihelion does not reveal the full structure of the atom velocity distribution as a function of ecliptic longitude as the interstellar distribution deforms in its path about the Sun. Understanding that structure is important in order to interpret the IBEX observations.

The purpose of this paper is to present a new analytical model for the heliospheric distribution of primary interstellar helium based on an expansion of the velocity distribution about its maximum value. Although this paper focuses on helium, the analysis presented applies to any interstellar atom species more massive than hydrogen with appropriate modification of the parameters. The new model shares with the previous model the restriction to ecliptic longitudes close to the sweet spot. However, the model includes spin axis tilts by small angles both in the ecliptic and out of the ecliptic. Section 2 presents the basic celestial mechanics, geometry and application of Liouville’s Theorem to derive the heliospheric helium distribution function; this material was derived in Lee et al. (2012) and is presented succinctly here as a basis for the subsequent mathematical manipulations. Section 3 simplifies the form of the distribution, calculates the velocity at peak intensity, and identifies the “direct” and “indirect” trajectories. Section 4 expands the logarithm of the distribution to second order in the atom velocity relative to the peak velocity to highlight the ellipsoidal structure of the distribution function as a function of longitude. The properties of the ellipsoid at the sweet spot are described in Section 5. The reduction of the distribution function due to photo-ionization is calculated in Section 6. Section 7 presents the heart of the analysis starting with integration of the atom differential intensity over atom speed and collimator solid angle to derive the theoretical prediction for the IBEX count rate. It includes the incorporation of spin axis tilt and the transformation of velocity coordinates to the Earth frame (approximately the IBEX frame), and ends with succinct analytical expressions for the count rate, the latitude of maximum count rate, the latitudinal variance, and the longitudinal variance of the maximum latitudinal count rate about the sweet spot. Section 8 highlights the effects of ionization and the integration over atom speed. Section 9 concludes with a summary of the results arising from this new expansion model for the distribution of interstellar helium in the heliosphere. The Appendix provides two alternate derivations of the latitude of peak count rate as a function of longitude, which ignore the integration over speed and collimator. They both treat the maximum value of the distribution function as surrogate measure of the count rate based on integration over speed and collimator. The first derivation involves expansion about the atom distribution function at perihelion, while the second involves a geometrical construct to calculate the velocity at the maximum value of the distribution subject to the constraint that the velocity lies in the plane normal to the spacecraft spin axis.

**We emphasize that the analysis we present relates strictly to the structure of primary interstellar helium in the heliosphere and the determination of its bulk velocity and temperature based on IBEX-Lo measurements. We do not address large-scale global models of the heliosphere (e.g., Izmodenov et al., 2015) or the helium secondary component known as the “Warm Breeze” (Bzowski et al., 2015). We refer the reader to Möbius et al. (2015), Bzowski et al. (2015) and McComas et al. (2015) for a historical perspective, the current best estimate of the helium parameters, and remaining questions.**

This development of an analytical model for the helium distribution in the heliosphere is part of a coordinated set of papers focused on the interstellar neutral gas as measured by IBEX; McComas et al. (2015) provides an overview of this Special Issue.

2. ATOM TRAJECTORIES AND THE DISTRIBUTION OF INTERSTELLAR HELIUM

The trajectory of an interstellar helium atom observed by IBEX is approximately scatter-free as it enters the heliosphere from the adjacent interstellar medium. Collisions with ions, electrons and atoms are rare and interactions with non-ionizing photons are inconsequential. There is a reduction of atom density due primarily to photo-ionization in the vicinity of the Sun, which we consider in Section 6. Therefore, the orbit of an interstellar helium atom of mass is a hyperbola with focus at the Sun. The orbital equation is given by



, (1)



where , is the mass of the Sun, is the gravitational constant, is the magnitude of the (constant) angular momentum, is the orbit eccentricity given by , is the total energy (constant) of the atom given by , and are the time-dependent polar coordinates of the atom about the Sun in the plane perpendicular to the angular momentum vector and passing through the center of the Sun, and increases from zero at perihelion as we follow the atom trajectory backwards in time. The fixed coordinate is the heliocentric radial distance of IBEX when the atom is detected, with speed and . We shall generally take , the average distance of Earth from the Sun, and neglect the eccentricity of Earth’s orbit and the distance of IBEX from Earth. Figure 1 shows the geometry of the orbit, where is the positive angle swept out by the atom as it proceeds from to the IBEX location at and . When , . Since and as , it is clear from equation (1) that . It is also clear from Figure 1 that .



These results may be combined to yield

. (2)



It is convenient to employ dimensionless variables and , where



and . In terms of these variables the eccentricity may be written as



, (3)



where . In terms of these variables equation (1) yields



. (4)



Combining equations (2) – (4), while requiring as evident in Figure 1 that if and if , we obtain



(5) . (6)  
Equations (1) – (6) identify the hyperbolic trajectory of a neutral helium atom in the gravitational field of the Sun, which is observed with coordinates and in its orbital plane in the frame of, and passing through, the Sun. The angle is defined by. This orbital plane must be placed in the heliospheric geometry appropriate to the varying position and viewing geometry of IBEX. Figure 2 shows the Cartesian geometry we invoke with generic position vector . The coordinates and measure the position in the ecliptic. The bulk velocity of the interstellar gas is given by



(7)



with . We take the longitude in the ecliptic to increase in the direction of Earth’s orbital motion and satisfy in the direction in which is anti-parallel to the ecliptic component of . We neglect both the eccentricity of Earth’s orbit, so that , and the difference between the location and velocity of Earth, and those of IBEX. The orbital plane of a detected atom contains the vector . The angle is the angle of rotation of the orbital plane about in the positive direction; implies that the orbital plane lies in the ecliptic. Finally, we note that our convention for may appear to be inconsistent with for ; although other authors have chosen the opposite convention, we have chosen for convenience.



According to Figures 1 and 2, an atom observed with coordinates has a velocity in interstellar space given by



. (8)



Since the motion of an atom is scatter-free, by Liouville’s Theorem the phase-space distribution function satisfies a version of the Vlasov equation and is conserved along the atom trajectory. Once the distribution in interstellar space is specified (assumed to be uniform and stationary), the distribution in the heliosphere is determined. We choose the interstellar distribution function to be a Maxwellian distribution drifting with velocity :



, (9)



where is Boltzmann’s constant, is helium atom density, and is temperature. The heliospheric distribution function of interstellar helium atoms observed with velocity at longitude is simply obtained by inserting equation (8) into equation (9). Defining dimensionless variables and , and noting that , we obtain for



. (10)



To illustrate the structure of the helium distribution derived and discussed in this paper we take the representative parameters (Möbius et al., 2015) , and . **While this temperature is not the most likely temperature for interstellar helium, it is the value chosen by Möbius et al. (2015) to highlight the comparison between this theoretical model and those of Sokol et al. (2015) and Schwadron et al. (2015).** The ecliptic longitude of the interstellar helium bulk velocity is generally defined by the direction toward which the gas flows. In terms of the standard definition of ecliptic longitude , we take as representative value for the ecliptic longitude of the helium gas in the local interstellar medium . The conversion in degrees between and is then . The reason for the term is the definition of in terms of the direction from which the gas flows. An objective of the comparison between this analytical model and the IBEX measurements is the determination of . With and , where is the mass of a proton, we obtain and . For future reference, we take the IBEX collimator to be conical with a half-angle aperture of (we describe the collimator response more precisely in Section 7), and the photo-ionization rate of helium at to be .



3. THE HELIUM DISTRIBUTION AS FUNCTION OF LATITUDE AND LONGITUDE

The argument of the exponential function in equation (10), , may be written as



, (11)



where the upper sign corresponds to , the lower sign to , and



, (12)  
 . (13)



Equation (11) may be further simplified to

, (14)  
where



, (15) . (16)



At the peak of the distribution function , which clearly requires  and either or . The latter conditions yield . The remaining condition that determines the peak of the distribution is for . Substitution of this value of into equation (16) and squaring both sides yields



, (17)



where the “plus” root is chosen to satisfy the obvious condition when . Equations (5) and (17) together implicitly determine , the value of at the peak for a specified longitude. Substituting equation (17) into equation (16) yields



. (18)



Since we have two cases. For , if , and if . For , if , and if . For a given there are two solutions corresponding to the peak of the “direct” trajectories with and the peak of the “indirect” trajectories with . They have the values of specified above. The peak values of and are given, respectively, by



(19) . (20)



The corresponding two peak values of for the direct and indirect trajectories are given by the implicit solution of equation (17). For parameters and , Figure 3 presents the curve as a function of in degrees, and horizontal lines with ordinate equal to for the specified values of in degrees. For a given value of , the intersection of the curve and the corresponding line for the smaller [larger] value of yields for the direct () [indirect ()] peak trajectory. The two values of merge for at the so-called “focusing cone,” where the direct and indirect orbits satisfy , and thus the atom intensity is enhanced. For two particular longitudes, , the direct peak trajectory satisfies , which implies that atoms at the peak are at perihelion. In this case equation (17) becomes



. (21)



Since IBEX measures atoms with velocities nearly perpendicular to its spin axis and the spin axis is oriented approximately toward the Sun, the atoms measured at these special longitudes are close to their perihelion. For the helium atoms the characteristic thermal speed divided by is given by for our representative parameters. When increases beyond this ratio, the measured intensity of helium decreases exponentially. Accordingly, the intensity of helium atoms should be greatest in a narrow range near . In fact, for IBEX there is a major distinction between the cases and . For Earth (and IBEX) measures the bulk helium with a speed of , whereas for Earth (and IBEX) measures the bulk helium with a speed of . Not only does the reduced speed decrease the intensity for , but also the ratio of the thermal speed to the helium bulk speed in the Earth’s frame is substantially larger so that the smaller intensity is spread over a greater angular measurement range. Indeed, interstellar helium has not been detected at (Galli et al., 2015), while the maximum helium intensity has been detected at , which is known within the IBEX Team as the “sweet spot.” Henceforth we consider only helium atoms measured in the vicinity of the sweet spot.



The helium atoms on indirect trajectories have velocity vectors not aligned with the IBEX collimator, and are substantially depleted by ionization (see Section 6 for a discussion of depletion by ionization). IBEX is not expected to detect indirect helium. Figure 4 illustrates Earth’s orbit and the direct and indirect bulk velocity trajectories of atoms observed at the sweet spot with . The trajectories are projected onto the ecliptic plane for the case . The angles , , and are indicated; we note that . We also note that the indirect trajectory is not closely aligned with the IBEX collimator and is close to the Sun during a portion of the trajectory. These features make observation of the indirect trajectory atoms by IBEX virtually impossible. Equation (17) yields an explicit approximate solution for characterizing the direct-trajectory peak at longitudes near the sweet spot where . We expand in equation (17) to second order in and obtain



, (22)



which is valid for sufficiently small values of . It is straightforward to expand to higher orders in powers of . In order to assess the range of for which equation (22) provides an accurate value of , we note that equation (17) yields an explicit solution for as a function of . Expanding to third order in , we generate three approximations to by first keeping only terms of first order in , then up to second order, and finally through third order. For each of these approximations, denoted by 1, 2 and 3, we subtract the exact solution and plot them in Figure 5 as a function of in the domain , which is larger than the domain of interest for IBEX measurements. The ordinate is an approximate measure of the accuracy of the expression for in degrees, . It is clear that the third order expression is excellent, the second order very good, and the first order is adequate. Accordingly, equation (22) (to second order) should suffice in the vicinity of the sweet spot where the measured intensities are most important. We shall return to this point in Section 7.



4. QUADRATIC EXPANSION OF THE HELIUM DISTRIBUTION FUNCTION ABOUT ITS PEAK

In order to calculate the basic structure of the helium distribution function about the direct trajectory peak, we expand the right-hand side of equation (10) about the direct peak to second order in . By construction only second-order derivatives survive. We obtain



.   
 (23)



The second-order terms that are linear in vanish identically. Evaluating the coefficients for corresponding to , we obtain



(24)



(25)



(26)



(27)



The expressions for , and are evaluated to first order in . The expression for is exact to all orders of . Equations (25) and (26) show that the variance of the velocity distribution in both and directions increases toward the focusing cone and decreases toward the inflow direction of the interstellar gas. The final term in equation (25) shows that the variance in the direction includes a quadratic term that increases away from the sweet spot; although that term was not calculated in equation (26), we presume that a similar term contributes to an increase in the variance away from the sweet spot in the direction as well. Equation (24) shows the opposite behavior of the velocity distribution in the direction of the gas beam, in which the distribution becomes narrower toward the focusing cone.



5. THE STRUCTURE OF THE HELIUM DISTRIBUTION AT THE SWEET SPOT

Equation (23) provides an approximate description of the helium distribution function near its peak in the inertial frame at a longitude near the sweet spot. It describes an ellipsoid centered at the peak, or bulk velocity, of the distribution. Defining the three orthogonal velocity components of , equation (23) becomes



, (28)



where and . If equation (28) is set equal to it describes an ellipsoidal surface, which defines the contour of the distribution whose value is . These coordinates are clearly not the principle-axis coordinates of . Only the coordinate is a principal-axis coordinate. Defining numbered axes and , which are rotated about the axis in the positive sense by angle , we obtain  
 (29)  
 . (30)  
We determine by requiring that the cross term involving vanish:  
 . (31)  
In these principal axis coordinates the equation for the family of ellipsoids is



. (32)  
For our choice of representative parameters for interstellar helium ( and ) we obtain from equations (24) – (27) for and to 3-figure accuracy , , and . Equation (31) then yields . From equation (32) we obtain the variances corresponding to as , and . It is clear from the variances that the interstellar thermal distribution has deformed at the sweet spot to an oval “pancake,” thinnest in a direction approximately in the ecliptic tilted from the bulk velocity and thickest in the direction approximately normal to the ecliptic. The deformation of the interstellar thermal distribution vanishes in the limit of large for which and we obtain . Figure 6 shows the geometry of the coordinate system introduced in this section and a schematic representation of the projection of the distribution function of onto the ecliptic at the sweet spot. The dashed lines show the principal axes of the ellipsoid in the projection plane. The ellipses are characteristic contours on a slice through the distribution parallel to the ecliptic. The rotation of the distribution about by is important and is discussed further in the Appendix .



The reduction in the width of the distribution in the direction of the bulk flow is due, in part, to the decrease of the gravitational potential energy of each particle. To illustrate this feature we consider 1-dimensional motion for which the potential energy per particle is and the particle distribution is a drifting Maxwellian with bulk flow velocity as . The distribution function is only a function of the total energy and is given by  
 , (33)  
where we consider only particles with . Differentiating equation (33) with respect to , we obtain the velocity of the distribution peak at as   
 . (34)



Differentiating a second time with respect to and setting , we obtain the variance as  
 , (35)  
where the final expression is given in dimensionless quantities. For the representative parameters quoted above we obtain , somewhat larger than the value quoted above. The term in equation (35) accounts for the term in equation (24), which clearly describes this effect. However, the next term in equation (24), , relates to the angular momentum of the atoms, which arrive at the sweet spot with the bulk velocity, through their orbital eccentricity . This effect, and the related tilt of the distribution, are not included in the 1-dimensional model, but are presumably also responsible for the difference between and .



6. DEPLETION DUE TO PHOTO-IONIZATION

Ionization of the helium on its journey through the heliosphere requires that we add a term to , the argument of the exponential function in equation (10), which describes the related depletion of neutral helium. Here, we assume that the ionization rates are stationary and proportional to (Lee et al., 2012):  
 , (36)  
where we have taken and expanded the general expression to first order in . Here , where is the photo-ionization rate at . For the purpose of this analysis, we neglect charge-exchange ionization and electron impact ionization, which are both small for helium. With we obtain . Expanding equation (36) about to first order in , we obtain  
   
 (37)  
The first term describes an overall decrease in the magnitude of the gas distribution. The second and third terms will be utilized in the next section. We shall show in Section 8 that the second and third terms contribute only very small corrections to the location and width of observed count rate peaks. However, the first term produces a substantial decrease in the magnitude of the helium distribution. For the standard parameters we have chosen, the exponential of this term yields a depletion factor for helium of 0.594.



We note that the expression for the reduction of the distribution function due to photo-ionization presented by Lee et al. (2012) is in error: the lead factor in their equation (25) should be replaced by . However, this error has no impact on equation (36) above since we have taken .



7. INTEGRATION OF ATOM INTENSITY OVER ENERGY AND COLLIMATOR SOLID ANGLE

The count rate in the IBEX frame, in which quantities are denoted by a prime, is given by the atom intensity integrated over speed (Lee et al., 2012) and the collimator acceptance in solid angle about the bore sight direction:  
.  
 (38)  
The quantity is the standard-deviation half-width of the conical collimator acceptance response, assumed for simplicity to extend to infinity in the angular deviation of the angles and from the IBEX bore sight direction, in which and . We define the bore sight direction to be in the same direction as the atom entering the collimator. The IBEX bore sight direction lies in the spin plane normal to the spacecraft spin axis. It makes an angle with the intersection of the spin plane and the ecliptic plane in the direction approximately antiparallel to Earth’s orbital motion (see Figure 2), which occurs because the IBEX spin axis is approximately Sun pointing. It should be emphasized that, in this configuration, a detected atom moves approximately anti-parallel to Earth’s orbital motion with . When the bore sight points into the northern hemisphere .



If the spin axis is Sun pointing, . However, the evaluation of the functions is complicated by the tilt of the IBEX spin axis. In general the evaluation proceeds as follows. Consider the triad of orthogonal unit vectors in the IBEX frame: parallel to the Earth’s orbital motion; in the direction of the Sun; and toward the N – pole perpendicular to the ecliptic. Let the spin axis of IBEX, , be defined by the two angles, and , as  
 . (39)  
We now construct another orthogonal triad of unit vectors in the IBEX frame: ,, and . Explicitly we obtain   
 . (40  
 . (41)  
A helium atom measured by IBEX (primed frame), ignoring the collimator width, must have a velocity in the IBEX scan plane given by  
 , (42)  
where corresponds to atoms whose trajectory lies in the ecliptic. Since , where is the radius vector from the Sun to IBEX, we have  
 , (43)  
which simplifies to  
 . (44)  
To first order in , and , we obtain  
 . (45)  
We note that the corrections to this expression are third order in , and , or smaller since in fact is generally small when the atom intensity is substantial. We neglect these higher-order effects. Similarly, is defined by , where includes the components of perpendicular to . We then obtain  
 . (46)  
Expanding equation (46) to second order in and , we obtain  
 . (47)



Equation (23) for includes terms up to second order in the angles and , and the difference in atom speed from that at the peak . These quantities are of the order of the ratio of the atom thermal speed to the atom bulk speed, which is , not to be confused with eccentricity. Accordingly, . The relevant spin-axis tilts satisfy . The collimator aperture satisfies and, if we restrict our attention to the vicinity of the sweet spot, we may also take . Since the IBEX angular measurement is not generally accurate to order , we shall be content with expressions accurate to order . However, one quantity of particular interest is the angle at which the count rate is largest. In order to calculate , the argument of the exponential function must be differentiated once with respect to . In order to calculate to order , we must include those terms of order in the expansion of , which include at least one factor of . In addition to these terms we include the terms in equation (37) describing photo-ionization that are linear in and even though the factor makes them ; we include them to assess in Section 8 the importance of ionization for helium. In addition we incorporate the factor into as and keep the first-order term in its expansion as . This term is also , but we retain it in order to assess its importance in Section 8.



Accordingly, equation (38) becomes  
 (48)  
  
where  
 (49)  
 , (50)  
and and are the second and third terms, respectively, of equation (37) describing ionization losses. Note that the constant terms in , including and the lead term in equation (37), are not kept since they only contribute to a constant multiplicative factor.



In order to perform the integrations over the primed variables, we must transform the variables appearing in equation (48) to primed variables. The Galilean transformation from the unprimed inertial-frame velocity variables to the primed local IBEX velocity variables of integration is  
 (51)  
 (52)  
 , (53)  
Equations (51) – (53) may be manipulated to yield  
 (54)  
 (55)  
 . (56)  
Since all four angles in equations (54) – (56) are small, the leading-order transformations determining are  
 (57)   
 (58)  
 . (59)  
The corrections to equations (57) – (59) are third order in the small angles and must be neglected since they add terms of higher order than those retained in equation (48). Thus, equations (57) – (59) may be utilized to transform the unprimed to the primed variables. In addition, for the term involving , the ratio must be expanded about as ; both terms must be kept since the correction term leads to a third-order term that must be kept as described above.



The final triple integral that must be evaluated is given by   
  
   
 , (60)  
where , and . The terms within the braces in equation (60) involving cubic terms in the integration variables apparently diverge exponentially at one endpoint of the integration. However, with the intention that these terms are small corrections, in the initial -integration we replace the exponential of the sum of these terms by . The resulting integration includes a factor of the form , which we then replace by for the subsequent integrations since and are of order unity. The final integrations (ignoring terms in the argument of the exponential in equation (60) of order ) yield  
  
 (61)  
where and . The lead terms within the braces are . The term involving is and must be neglected since terms of the same order in the original expansion have been discarded. The functions and depend on ; we shall return to them in more detail soon. It is clear that equation (61) describes an energy-integrated slice through the helium distribution. Its maximum occurs when during its scan IBEX measures atom velocities that are closest to the peak (first term), and the value of that maximum depends on how close to the actual distribution peak the scan plane is oriented (second term). Clearly the count rate depends on and through , which describes the orientation of the IBEX scan plane. It also depends on ionization rates through and , and the measurement of atom intensity through , both of which modify the structure of the energy-integrated distribution.



By differentiating the argument of the exponential function in equation (61) and setting it equal to zero, we obtain the latitude during the IBEX scan, , at which the maximum intensity is measured. Besides differentiating and explicitly, it turns out that the dominant term of the derivative of arises from   
 . (62)   
With this in mind we obtain  
 , (63)  
where the distinction between and may be ignored in this expression, accurate to .



We now evaluate as a function of standard ecliptic longitude . As stated in Section 2 the conversion in degrees is . Figure 7 presents in degrees based on equation (63) as a function of ecliptic longitude for our standard set of helium parameters with and for both and . The solid curves evaluate based on the lead term in equation (22), which is consistent with the accuracy of equation (63). However, since it is evident in Figure 4 that becomes larger than the -ordering implies for the largest and smallest longitudes included in Figure 7, the dashed curves include both terms of equation (22) to evaluate in equation (63). The dashed curves are in good agreement with the observed values of extrapolated to (Möbius et al., 2015). Figure 8 presents in degrees based on equation (63) over a reduced range of ecliptic longitude for with (solid lines), and with (dashed lines), in both cases for a range of characteristic of IBEX spin axis drift. It is clear that is only important for larger values of .



Differentiating equation (61) twice with respect to we obtain the variance of the latitude distribution in the IBEX frame as  
 , (64)  
where the lead terms are and the correction comes from the second term in braces in equation (61) and higher order terms in the Galilean transformation. As mentioned in Section 4, may be written exactly rather than as a power series in or . Although this is not entirely consistent with our expansion procedure, we insert the exact expression into equation (64):   
 . (65)  
Note that formally the factor should be replaced by unity since we are neglecting terms of the same order. However, we shall keep it since the expression for is exact. The factor is of course critical as . We note that in the region where IBEX measures the largest helium intensities. Figure 9 plots as presented in equation (65) in degrees as a function of ecliptic longitude for our standard set of parameters. This prediction is consistent with the latitudinal widths of the helium distribution measured by IBEX (Möbius et al., 2015).



We now calculate the variance in longitude of the count rate measured at the peak in latitude as a function of longitude. Substituting equation (63) into equation (61), we obtain   
, (66)  
where the contribution from the first term in braces in equation (61) is of higher order and is therefore neglected. The last term in square brackets in equation (66) is also one order of higher than the first terms. Retaining only the lead terms we obtain  
 . (67)  
The variance follows as  
 , (68)   
where is evaluated at . It is clear that the spin-axis tilt in the ecliptic, measured by , shifts the “sweet spot.” The result expressed in equation (67) may be somewhat misleading since in the absence of spacecraft reorientation maneuvers , where is the longitude at which the spin axis is sunward pointing in the ecliptic. Since , may certainly be modified from the form given in equation (68). For the standard set of parameters we have chosen, we obtain . Multiplying by the conversion factor to full-width-half-maximum (FWHM), , yields .



In order to interpret the analytical results it is useful to visualize the helium distribution function in both inertial and IBEX frames and to consider the slice through it, within which IBEX-Lo measures an energy- and collimator-integrated count rate for each angle about the IBEX spin axis. Figures 10 and 11 show schematic projections of the velocity distribution into the , or , plane (Fig. 10) and into the , or , plane (Fig. 11); recall the velocity variables introduced in Section 5 and Figure 5. The ellipses are contours on a slice through the distribution parallel to the projection plane. Figure 10 shows a case for (longitudes greater than the sweet spot) while Figure 11 shows a case for (longitudes less than the sweet spot). In Figure 10 the orbital velocity of Earth , the direction to the Sun , the spin axis direction, the velocity coordinate unit vector , and the IBEX collimator field-of-view (FoV) are indicated. The coordinates of the distribution peak, and , located at the intersection of the dashed lines, are indicated. The intersection of the IBEX spin plane (approximately within which velocity measurements are made) with the plane passing through the distribution peak is indicated; its tilt and abscissa are determined by as shown. It is clear that the peak of the distribution within the spin plane has a different coordinate than the actual peak of the distribution. Thus a different value of is observed, which depends on . Figure 11 shows the projection of velocity space in the Earth’s frame into the ecliptic plane. Velocities , and angles , in the inertial (Sun) and Earth frames, as well the IBEX spin axis drift tilt , are indicated. The tilt of the distribution contours relative to the - direction was described in Section 5, and will be discussed further in the Appendix.



**In the Appendix we present two alternative derivations of equation (63) for the peak latitude as a function of longitude. Even though both ignore the integration over energy and collimator, with the appropriate addition of both give the same result. The derivations are instructive for different reasons. The first, which expands about (describing atoms at perihelion) and the speed of the maximum value of the phase-space density, is the less complete expansion used by Leonard et al. (2015) [see Figure 12 associated with the discussion in the first full paragraph following equation (A.11) in the Appendix]. The second is a geometrical approach, which is instructive in helping to visualize how results from the shape of the distribution ellipsoid and the orientation of the spin plane.**



8. EFFECTS OF IONIZATION AND ATOM INTENSITY

It is clear from the structure of equations (61) and (63) that the quantities arising from the phase-space and intensity integration, , and the effects of photo-ionization, and , simply add terms to the quantity in equation (63). Although these terms are , and therefore comparable with many terms describing other features of the helium distribution that we have neglected, it is interesting to calculate their effect on the latitudinal peak . Evaluating and from equation (37), we obtain the terms that add to in equation (63) due to photo-ionization and the speed integration:  
. (69)  
For our standard choice of parameters, the lead term within the braces, which describes the effect of the intensity integration, effectively adds to in equation (63) an angle of . Similarly, the remaining terms in braces describe the effect of photo-ionization and add to an angle of . Since is characteristic of the small angles we have considered, which corresponds to radians, an angle of order corresponds to and is somewhat smaller than these effects. The effective increase of for both effects is understandable: both the intensity integration and photo-ionization enhance the importance of the higher energy atoms in the distribution, which are slightly less deflected by solar gravity and are therefore characterized by larger .



9. SUMMARY AND CONCLUSIONS

We have presented a new analytical model for the distribution of interstellar helium in the heliosphere. The underlying structure of the model was derived by Lee et al. (2012) and is based on Liouville’s Theorem, the assumption of a homogeneous drifting Maxwellian distribution at large distances (in local interstellar space), and stationary spherically-symmetric ionization rates. This structure establishes the velocity distribution in the inertial frame of the Sun. Using an expansion about the peak of the distribution to second-order in inverse powers of the gas Mach number and/or the deflection of the atom trajectories from alignment with Earth’s orbital velocity about the Sun, we have calculated the ellipsoidal shape of the distribution. The ellipsoid is elongated in the direction normal to the ecliptic and compressed in a direction in the ecliptic that is deflected sunward from the direction of the bulk flow by about .



We have then calculated the predicted count rate of IBEX by transforming the distribution function velocity variables to the IBEX frame and integrating the helium atom differential intensity over energy and the collimator aperture solid angle. An important generalization of the results of Lee et al. (2012) is the inclusion of the tilt of the IBEX spin axis by small angles out of the ecliptic and in the ecliptic. Since the IBEX-Lo detector bore sight scans perpendicular to the spin axis, a tilt results in measurements through a different slice of the distribution function, which must be taken into account in order to predict the bulk velocity and temperature of the interstellar distribution. Tilting the spin axis in a southward direction allows a more accurate determination of the ecliptic longitude of the interstellar bulk velocity relative to the Sun (Möbius et al., 2012). The tilt of the spin axis in the ecliptic occurs naturally as the Earth and IBEX orbit the Sun after the reorientation of the spin axis at the beginning of each orbit arc.



The expansion in the observed count rate about the peak of the distribution, under limitation to longitudes near the overall peak, which occurs when the helium bulk velocity is nearly parallel to the IBEX bore sight and Earth’s orbital velocity, provides surprisingly simple analytical expressions. These expressions, in principle, allow us to derive sequentially the interstellar helium bulk velocity and temperature from the measured quantities, as described in Möbius et al. (2015).

We now outline that sequence. As mentioned in Section 2, the variable is defined in terms of standard ecliptic longitude by the relation , where is the ecliptic longitude of the interstellar helium bulk flow. Although we state the relation in radians, ecliptic longitude is generally expressed in degrees. From the squared factor within square brackets in equation (67), the sweet spot expressed in ecliptic longitude is given by  
 , (70)  
where is the ecliptic longitude of maximum count rate as measured at . However, this expression must be modified. The tilt of the spin axis in the ecliptic plane is naturally due to the drift of the spin axis as the Earth (and IBEX) moves in its orbit. In the absence of a reorientation of the spin axis, we obtain  
 , (71)  
where is the ecliptic longitude at which the ecliptic component of the spin axis points sunward. Substituting equation (71) into equation (70), we obtain  
 . (72)  
This relation determines the sweet spot in ecliptic coordinates in terms of . Substituting equation (72) into equation (71) we obtain  
 , (73)  
which provides a direct relationship between and . The dependences on and on through provide only higher order corrections. Measuring at the sweet spot yields via equation (63). Measuring the latitudinal variance at the sweet spot gives via equation (64). Finally, comparing observations of the latitudinal peak as a function of ecliptic longitude with equation (63) yields . This sequential scheme to deduce the interstellar parameters based on IBEX observations was introduced in Möbius et al. (2012).



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**APPENDIX: TWO ALTERNATIVE DERIVATIONS FOR PEAK LATITUDE AS FUNCTION OF LONGITUDE**

**It is instructive to consider two simpler but less complete alternative derivations of the peak latitude as a function of observer longitude, which ignore the integration over both energy and the collimator solid angle. The first** derivation was in fact used in the analysis presented in Leonard et al. (2015). The derivation is based on the expansion of not about the peak of the velocity distribution at in inertial frame velocity coordinates, but rather about , where is the speed at which has its maximum value at the observer location. The nearly sunward pointing of the IBEX spin axis, which dictates , motivates the expansion about .



The value of as a function of longitude is given implicitly by equation (41) or (44) in Lee et al. (2012). If , then may be obtained by differentiating equation (23) with respect to , subject to and , and setting the result equal to zero. The result is   
 . (A.1)   
It can be shown explicitly that the implicit equations in Lee et al. (2012) reduce to equation (A.1) when , or equivalently , is satisfied.



The alternative expansion of yields  
, (A.2)  
where we use the notation[evaluated at ] and by construction. The higher-order terms are neglected since they only contribute to at or higher. Transforming equation (A.2) to IBEX coordinates using equations (57) – (59), we obtain  
  
 , (A.3)  
where is the speed at which is maximized for [see equation (45)], which will now be determined. The peak value of occurs at and , which are specified by  
 (A.4)  
   
 . (A.5)  
Equation (A.4) yields  
 . (A.6)  
Substituting equation (A.6) into equation (A.5) yields  
 . (A.7)  
The factors, , and are each of order . Since we are only attempting accuracy to and both and are , there is no need to distinguish between evaluating and its derivatives at and at . Accordingly, , , , and , where these values of may be evaluated at the sweet spot, . Similarly, the ratio following in equation (A.7) may be replaced by . However, must retain its dependence on , which is given by equation (A.1). After some algebra, equation (A.7) then becomes  
 . (A.8)  
For the expression in square brackets, we must distinguish between and . Expanding that expression to first order in , equation (A.8) becomes  
 . (A.9)  
With the exception of the collimator solid angle, the right-hand side of equation (A.9) is identical to the last term in equation (63). Combining equations (A.1) and (A.6) yields   
 . (A.10)  
Expanding the term, , in equation (A.9) to first order in , and substituting equation (A.10), yields  
 , (A.11)  
which is accurate to and is identical to equation (63) if . The fact that the two expressions agree implies that, to order , is independent of our choice to either integrate over speed or to evaluate the distribution function at the speed at its maximum value. This result is consistent with the finding in Section 8 that the intensity integration introduces corrections of order .



Equation (A.11) does not agree with equation (4) in Leonard et al. (2015) for , the location of the peak in latitude, even though both are based on expansion about . The formula in Leonard et al. (2015) took the dependence of to arise from the factor in the last term in brackets in equation (A.7). This term, however, is of order and is neglected in equation (A.11). The second-order terms involving in equation (A.11) arise from the first term in square brackets in equation (A.7), which was ignored in the formula in Leonard et al. (2015), and from the choice of (as opposed to ) in equation (A.9) as the speed which controls the Galilean transformation. For , equation (A.11) is identical with the formula in Leonard et al. (2015). However, Leonard et al. (2015) use the dependence of to estimate the value of that would be observed had the spin axis pointed toward the Sun (or ). That estimate should be affected by the larger second-order terms that depend on and are evident in equation (A.11). Figure 12 shows the difference between equation (A.11) [actually equation (63), which includes the collimator aperture in solid angle] (solid lines) and the expression used for in Leonard et al. (2015) (dashed lines) for and as a function of ecliptic longitude. It is clear that the dependences on are very different. Nevertheless, it is clear from Figure 7 that these differences are small for the small values of featured in the work of Leonard et al. (2015).



**The second alternative derivation to determine the spin angle of maximum intensity as a function of ecliptic longitude, , while ignoring integration over speed and the entrance aperture, is based on a geometrical approach.** We obtain the IBEX spin axis from equation (37) to as  
 , (A.12)  
where the unit vectors following the last equality are defined at the beginning of Section 5. In the IBEX frame, the velocity of a helium atom is given by  
 . (A.13)  
Ignoring the solid angle of the IBEX aperture, IBEX only detects atoms that satisfy , which yields the constraint  
 . (A.14)



Writing , which describes the helium atom distribution ellipsoid given by equation (23), in IBEX coordinates using equations (57) – (59), we obtain  
   
 . (A.15)  
Rewriting the angles in terms of the IBEX velocity coordinates and using constraint (A.14) to eliminate , equation (A.15) yields  
  
 . (A.16)  
The peak value of the distribution in -space within the plane in velocity space specified by equation (A.14) is obtained by differentiating equation (A.16) with respect to and , and setting the two partial derivatives equal to zero. To first order in we obtain at the peak as  
 , (A.17)  
where may be replaced by at this order. Similarly, to second order in , we obtain at the peak as  
 , (A.18)   
where again has been replaced by at this order. Substituting equation (A.17) into equation (A.18), and noting that and , we obtain  
 . (A.19)  
Expanding and substituting equation (A.17), we obtain finally  
 . (A.20)  
Equation (A.20) is identical to equation (A.11). This derivation involves an appealing geometrical analysis to identify the peak of the distribution in the spin plane accessible to the IBEX detector. Figure 10 illustrates approximately the essence of this derivation: where the spin plane is tangent to the helium distribution contour line determines the speed and angle of the distribution maximum accessible to measurement by IBEX-Lo for specified values of .



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FIGURE CAPTIONS

**Figure 1**. Hyperbolic trajectory of an atom with polar coordinates and (measured in the positive sense from perihelion) in the plane passing through IBEX and the center of the Sun, and normal to its angular momentum. As the atom velocity is ; at IBEX its coordinates are and , its velocity is , and its speed is . At IBEX the atom has swept out the angle ; if IBEX happens to observe the atom at perihelion, it has swept out the angle . This figure is a revised version of Figure 1 in Lee et al. (2012).



**Figure 2**. Geometry governing IBEX measurements of the interstellar gas. The ecliptic defines the x – y plane with the x-direction defined to be parallel to the projection of , the bulk velocity of the interstellar gas, onto the ecliptic. The angle is defined by with . Earth’s orbit is assumed to be circular. The position vector of IBEX/Earth is , Earth’s velocity is , and ecliptic longitude is measured from the negative x-axis increasing in the direction of Earth’s orbital motion. The vector describes the components of atom velocity normal to . The angle defines the tilt of the plane containing the atom orbit about the vector in the positive sense. This figure was published in Lee et al. (2012).



**Figure 3**. The curve is a plot of given by equation (5) as a function of in degrees for , , and . The horizontal lines have ordinate equal to for and the specified values of in degrees. The intersection of the line for a specified with the curve for the smaller (larger) value of yields for the direct (indirect) trajectory observed at that .



**Figure 4**. Direct and indirect trajectories of the interstellar bulk velocity intersecting Earth orbit at the sweet spot for and . The values of and are indicated for each trajectory at the sweet spot, at which the direct trajectory satisfies .



**Figure 5**. Plots of as a function of (expressed in radians) based on expansions of equation (17) to first (1), second (2) and third (3) order in powers of . The exact solution valid to all orders has been subtracted from each expanded approximate solution. The range of covers the range of IBEX measurements sufficiently close to the sweet spot that they are of importance for the analysis of interstellar helium. The ordinate, , gives the approximate error (in degrees) for the determination of . The accuracy of the third-order expansion is extremely good. The accuracy of the first-order and second-order expansions indicates that the expansion of in powers of given by equation (22) is sufficient for the expressions calculated in Section 7.



**Figure 6**. Schematic projection of the helium velocity distribution about its peak value at the sweet spot onto the plane, which is approximately parallel to the ecliptic in coordinate space. The contours illustrate the ellipsoidal structure of the distribution. The dashed lines represent the principal axis directions and , rotated as shown by . The Sun-pointing unit vector is ; Earth’s orbital velocity is .



**Figure 7**. Latitude of the peak count rate, , as a function of ecliptic longitude, , based on equation (63). The plots are evaluated for the chosen set of representative parameters, , and the two indicated values of . The solid (dashed) curves are based on the first term (first and second terms) of equation (22) to evaluate .



**Figure 8**. Latitude of the peak count rate, , as a function of ecliptic longitude, , based on equation (63) and evaluated for our chosen set of representative parameters. The solid curves are for and as indicated; the dashed curves are for and as indicated.



**Figure 9**. Latitudinal variance of the count rate, , as a function of ecliptic longitude, , based on equation (65) and our chosen set of representative parameters.



**Figure 10**. Schematic slice through the peak of the helium velocity distribution in the plane for a case with . The contours illustrate the elliptical structure of the distribution and the dashed lines show the projection of the principal axis directions onto the plane. The peak of the distribution has velocity coordinates . The spin axis has a tilt out of the ecliptic as shown, and a tilt in the ecliptic so that the IBEX spin plane intersects the plane of the figure along the line indicated. The field-of-view (FoV) of the collimator beyond the spin plane is indicated schematically. The point where the contour is tangent to the spin plane determines the scan latitude that results in the maximum count rate as indicated. This schematic representation is approximate since the point of tangency does not precisely occur in the plane of the figure.



**Figure 11**. Schematic slice through the peak of the helium velocity distribution parallel to the ecliptic plane for a case with . The contours illustrate the elliptical structure of the distribution and the dashed line shows one of the principal axis directions in the plane, tilted relative to . The IBEX spin plane intersects the plane of the figure along the line indicated for and as indicated. The field-of-view (FoV) of the collimator beyond the spin plane is indicated schematically. The peak velocity and the -coordinate are in the frame of the Sun and in the IBEX/Earth frame. The Sun-pointing unit vector is ; Earth’s orbital velocity is .



**Figure 12**. Latitude of the peak count rate, , as a function of ecliptic longitude, , for our chosen set of representative parameters. The solid curves are based on equation (63) for and as indicated, and as presented in Figure 8. The dashed curves are based on the incorrect expression for , used by Leonard et al. (2015) and described in the paragraph where this figure is referenced, for the same set of parameters, , and as indicated.

